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Quasi-energy function for diffeomorphisms with wild separatrices

V. Grines* F. Laudenbach† O. Pochinka‡

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Abstract

According to Pixton [8] there are Morse-Smale diffeomorphisms of $\mathbb{S}^3$ which have no energy function, that is a Lyapunov function whose critical points are all periodic points of the diffeomorphism. We introduce the concept of quasi-energy function for a Morse-Smale diffeomorphism as a Lyapunov function with the least number of critical points and construct a quasi-energy function for any diffeomorphism from some class of Morse-Smale diffeomorphisms on $\mathbb{S}^3$.

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1 Formulation of results

According to [3], given a closed smooth n-manifold M^n and a Morse function $\varphi : M^n \to \mathbb{R}$ is called a *Morse-Lyapunov function* for Morse-Smale diffeomorphism $f : M^n \to M^n$ if:

1) $\varphi(f(x)) < \varphi(x)$ if $x \notin Per(f)$ and $\varphi(f(x)) = \varphi(x)$ if $x \in Per(f)$, where $Per(f)$ is the set of periodic points of f;

2) any point $p \in Per(f)$ is a non-degenerate maximum of $\varphi|_{W^u(p)}$ and a non-degenerate minimum of $\varphi|_{W^s(p)}$.

Definition 1.1 *Given a Morse-Smale diffeomorphism $f : M^n \to M^n$, a function $\varphi : M^n \to \mathbb{R}$ is a quasi-energy function for f if φ is a Morse-Lyapunov function for f and has the least possible number of critical points among all Morse-Lyapunov functions for f.*

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In this paper we consider the class G_4 of Morse-Smale diffeomorphisms $f : \mathbb{S}^3 \to \mathbb{S}^3$ whose nonwandering set consists of exactly four fixed points: one source α, one saddle σ and two sinks ω_1 and ω_2. It follows from [9] (theorem 2.3), that the closure of each connected component (separatrix) of the one-dimensional manifold $W^u(\sigma) \setminus \sigma$ is homeomorphic to a segment which consists of this separatrix and two points: σ and some sink. Denote by ℓ_1, ℓ_2 the one-dimensional separatrices containing the respective sinks ω_1, ω_2 in their closures. According to [9], $\bar{\ell}_i, i = 1, 2$ is everywhere smooth except, maybe, at ω_i. So the topological embedding of $\bar{\ell}_i$ may be complicated in a neighborhood of the sink.

According to [1], ℓ_i is called *tame* (or *tamely embedded*) if there is a homeomorphism $\psi_i : W^s(\omega_i) \to \mathbb{R}^n$ such that $\psi_i(\omega_i) = O$, where O is the origin and $\psi_i(\bar{\ell}_i \setminus \sigma)$ is a ray starting from O. In the opposite case ℓ_i is called *wild*. It follows from a criterion in [4] that the tameness of ℓ_i is equivalent to the existence of a smooth 3-ball B_i around ω_i in any neighborhood of ω_i such that $\ell_i \cap \partial B_i$ consists of exactly one point. Using lemma 4.1 from [3] it is possible to make this criterion more precise in our dynamical setting: ℓ_i is tame if and only if there is 3-ball B_{ω_i} such that $\omega_i \in f(B_{\omega_i}) \subset int\ B_{\omega_i} \subset W^s(\omega_i)$ and $\ell_i \cap \partial B_{\omega_i}$ consists of exactly one point.

It was proved in [2] that, for every diffeomorphism $f \in \mathcal{G}_4$, at least one separatrix (ℓ_1 say) is tame. It was also shown that the topological classification of diffeomorphisms from $\mathcal{G}_4$ is reduced to the embedding classifications of the separatrix ℓ_2; hence there are infinitely many diffeomorphisms from $\mathcal{G}_4$ which are not topologically conjugate.

To characterize a type of embedding of ℓ_2 we introduce some special Heegaard splitting of $\mathbb{S}^3$. Let us recall that a three-dimensional orientable manifold is *a handlebody of genus* $g \geq 0$ if it is obtained from a 3-ball by an orientation reversing identification of g pairs of pairwise disjoint 2-discs in its boundary. The boundary of such a handlebody is an orientable surface of genus g.

Let $P^+ \subset \mathbb{S}^3$ be a handlebody of genus g such that $P^- = \mathbb{S}^3 \setminus intP^+$ is a handlebody (necessarily of the same genus as P^+). Then the pair (P^+, P^-) is a Heegaard splitting of genus g of $\mathbb{S}^3$ with Heegaard surface $S = \partial P^+ = \partial P^-$.

Definition 1.2 *A Heegaard splitting (P^+, P^-) of $\mathbb{S}^3$ is said to be adapted to $f \in G_4$, or f-adapted, if:*

a) $\overline{W^u(\sigma)} \subset f(P^+) \subset int\ P^+$;

b) $W^s(\sigma)$ intersects ∂P^+ transversally and $W^s(\sigma) \cap P^+$ consists of a unique 2-disc.

An f-adapted Heegaard splitting $\mathbb{S}^3 = P^+ \cup P^-$ is said to be minimal if its genus is minimal among all f-adapted splittings.

For each integer $k \geq 0$ we denote by $G_{4,k}$ the set of diffeomorphisms $f \in G_4$ for which the minimal f-adapted Heegaard splitting has genus k. It is easily seen that, for each $f \in G_{4,0}$, ℓ_2 is tame and, according to [3], f possesses an energy function. Conversely any diffeomorphism in $G_{4,k}$, $k > 0$, has no energy function (see [8]). Figure 1 shows the phase portrait of a diffeomorphism $G_{4,1}$. The main result of this paper is the following.

Theorem 1 *Every quasi-energy function for a diffeomorphism $f \in G_{4,1}$ has exactly six critical points.*

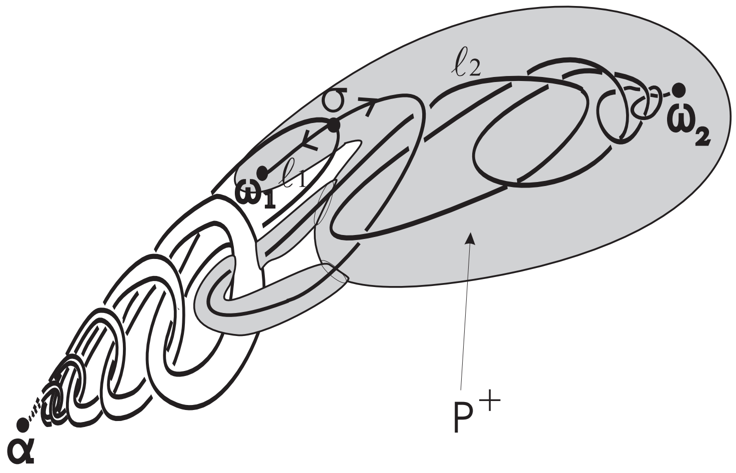

Figure 1: A diffeomorphism from the class $G_{4,1}$

2 Recollection of Morse theory

According to Milnor ([6], section 3), we use the following definitions.

A compact $(n+1)$-dimensional *cobordism* is a triad (W, L_0, L_1) where L_0 and L_1 are closed manifolds of dimension n and W is a compact $(n+1)$-dimensional manifold whose boundary consists of the disjoint union $L_0 \cup L_1$. It is an *elementary* cobordism when it possesses a Morse function $\varphi : W \to [0,1]$ with only one critical point and such that $\varphi^{-1}(i) = L_i$ for $i = 0, 1$. When the index of the unique critical point is r, one speaks of an elementary cobordism of index r.

In this situation, L_1 is obtained from L_0 by a *surgery* of index r, that is: there is an embedding $h : \mathbb{S}^{r-1} \times \mathbb{D}^{n-r+1} \to L_0$ such that L_1 is diffeomorphic to the manifold obtained from L_0 by removing the interior of the image of h and gluing $\mathbb{D}^r \times \mathbb{S}^{n-r}$, or

$$L_1 \cong \mathbb{D}^r \times \mathbb{S}^{n-r} \bigcup_{h|_{\mathbb{S}^{r-1} \times \mathbb{S}^{n-r}}} L_0 \setminus int\ (h(\mathbb{S}^{r-1} \times \mathbb{D}^{n-r+1}))\,.$$

Conversely, the following statement holds (see [6], Theorem 3.12):

Statement 2.1 *If L_1 is obtained from L_0 by a surgery of index r, then there exists an elementary cobordism (W, L_0, L_1) of index r.*

On figure 2 it is seen a surgery of index 1 from the 2-sphere to the 2-torus with some level sets of a Morse function on the corresponding elementary cobordism.

Finally, we recall the weak Morse inequalities (see [5], Theorem 5.2).

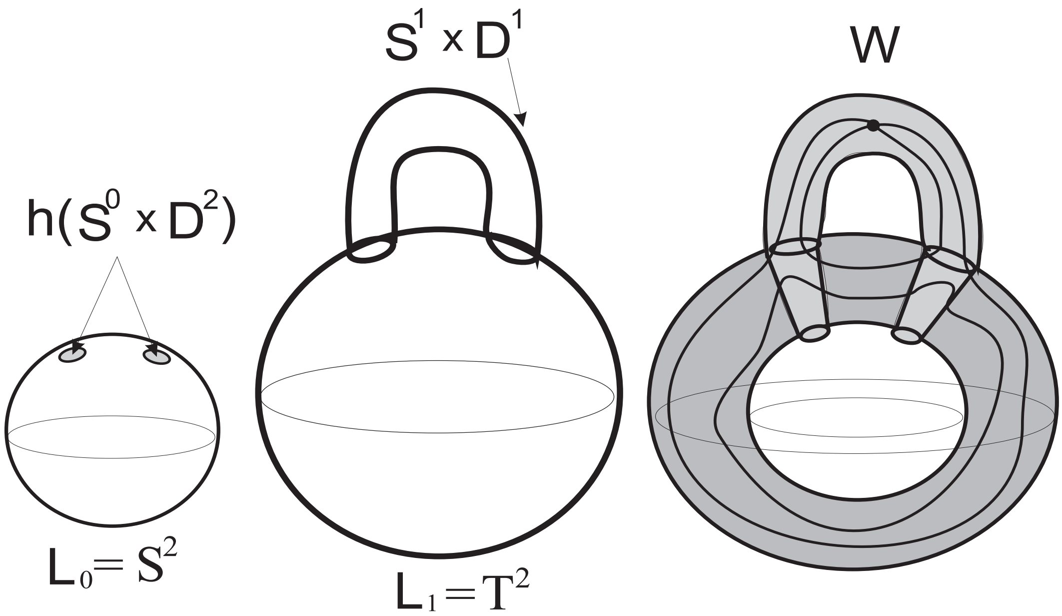

Figure 2: An elementary cobordism

Statement 2.2 *Let M^n be a closed manifold, $\varphi : M^n \to \mathbb{R}$ be a Morse function, C_q be the number of critical points of index q and $\beta_q(M^n)$ be the q-th Betti number of the manifold M^n. Then $\beta_q(M^n) \leq C_q$ and the Euler characteristic $\chi(M^n) := \sum\limits_{q=0}^{n}(-1)^q\beta_q(M^n)$ equals $\sum\limits_{q=0}^{n}(-1)^qC_q$.*

3 Proof of Theorem 1

Let f be a Morse-Smale diffeomorphism of the 3-sphere belonging to $G_{4,1}$. As the number of critical points of any Morse function on a closed 3-manifold is even (it follows from statement 2.2) and greater than four (as $Per(f) \subset Cr(\varphi)$ and ℓ_2 is wild) then, for proving theorem 1, it is enough to construct a Lyapunov function with six critical points.

3.1 Auxiliary statements

For the proof of the following statements 3.1 and 3.2 we refer to [3], lemma 2.2 and lemma 4.2.

Statement 3.1 *Let p be a fixed point of a Morse-Smale diffeomorphism $f : M^n \to M^n$ such that $\dim W^u(p) = q$. Then, in some neighborhood U_p of p, there exist local coordinates $x_1, \ldots, x_n$ vanishing at p and an energy function $\varphi_p : U_p \to \mathbb{R}$ such that*

$$\varphi_p(x_1, \ldots, x_n) = q - x_1^2 - \ldots - x_q^2 + x_{q+1}^2 + \ldots + x_n^2$$

and $(TW^u(p) \cap U_p) \subset Ox_1 \ldots x_q$, $(TW^s(p) \cap U_p) \subset Ox_{q+1} \ldots x_n$.

Statement 3.2 *Let ω be a fixed sink of a Morse-Smale diffeomorphism $f : M^3 \to M^3$ and B_ω be a 3-ball with boundary S_ω such that $\omega \in f(B_\omega) \subset int\ B_\omega \subset W^s(\omega)$. Then there exists an energy function $\varphi_{B_\omega} : B_\omega \to \mathbb{R}$ for f having S_ω as a level set.*

Lemma 3.3 *Let ω be a fixed sink of a Morse-Smale diffeomorphism $f : M^3 \to M^3$ and Q_ω be a solid torus such that $\omega \in f(Q_\omega) \subset int\ Q_\omega \subset W^s(\omega)$. Then there exists a 3-ball B_ω such that $f(Q_\omega) \subset B_\omega \subset int\ Q_\omega$.*

Proof: Let D_0 be a meridian disk in Q_ω such that $\omega \notin D_0$. As $Q_\omega \subset W^s(\omega)$ there is an integer N such that $f^n(Q_\omega) \cap D_0 = \emptyset$ for every $n > N$. We may also assume that D_0 is transversal to $G = \bigcup\limits_{n \in \mathbb{Z}} f^n(\partial Q_\omega)$, and hence $G \cap int\ D_0$ consists of a finite family $\mathcal{C}_{D_0}$ of intersection curves. Each intersection curve $c \in \mathcal{C}_{D_0}$ belongs to $f^k(\partial Q_\omega)$ for some integer $k \in \{1, \dots, N\}$. There are two cases: (1) c bounds a disk on $f^k(\partial Q_\omega)$; (2) c does not bound a disk on $f^k(\partial Q_\omega)$. Let us decompose $\mathcal{C}_{D_0}$ as union of two pairwise disjoint parts $\mathcal{C}^1_{D_0}$ and $\mathcal{C}^2_{D_0}$ consisting of curves with property (1) or (2), accordingly.

Let us show that there is a meridian disk D_1 in Q_ω such that D_1 is transversal to G and $G \cap int\ D_1$ consists of family $\mathcal{C}_{D_1} = \mathcal{C}^2_{D_0}$ of intersection curves. If $\mathcal{C}^1_{D_0} = \emptyset$ then $D_1 = D_0$. In the opposite case for any curve $c \in \mathcal{C}^1_{D_0}$ denote by d_c the disk on $f^k(\partial Q_\omega)$ such that $\partial d_c = c$. Notice that d_c does not contain a curve from the family $\mathcal{C}^2_{D_0}$. Then there is $c \in \mathcal{C}_{D_1}$ which is innermost on $f^k(\partial Q_\omega)$ in the sense that the interior of d_c contains no intersection curves from $\mathcal{C}_{D_0}$. For such a curve c denote e_c the disk on D_0 such that $\partial e_c = c$. As $int\ Q_\omega \setminus D_0$ is an open 3-ball then $e_c \cup d_c$ bounds a unique 3-ball $b_c \subset int\ Q_\omega$. Set $D'_c = (D_0 \setminus e_c) \cup d_c$. There is a smooth approximation D_c of D'_c such that D_c is a meridian disk on Q_ω, D_c is transversal to G. Moreover $G \cap int\ D_c$ consists of a family $\mathcal{C}_{D_c}$ of intersection curves having less elements than $\mathcal{C}_{D_0}$; indeed, c disappeared and also all curves from $\mathcal{C}_{D_0}$ lying in $int\ e_c$. We will repeat this process until getting a meridian disk D_1 with the required property.

Now let $c \in \mathcal{C}_{D_1}$, $c \in f^k(\partial Q_\omega)$. Denote e_c the disk that c bounds in D_1. Let us choose c innermost in D_1 in the sense that the interior of e_c contains no intersection curves from $\mathcal{C}_{D_1}$. There are two cases: (a) $e_c \subset f^k(Q_\omega)$ and (b) $int\ e_c \cap f^k(Q_\omega) = \emptyset$.

In case (a) e_c is a meridian disk of $f^k(Q_\omega)$ and $D = f^{-k}(e_c)$ is a meridian disks in Q_ω such that $f(Q_\omega) \cap D = \emptyset$. Indeed, by construction $int\ e_c \cap G = \emptyset$, hence $int\ D \cap G = \emptyset$. Thus we can find the required 3-ball B_ω inside $int\ Q_\omega \setminus D_1$.

In case (b) there is a tubular neighborhood $V(e_c) \subset int\ Q_\omega$ of the disk e_c such that $G \cap int\ V(e_c) = \emptyset$ and $B_k = f^k(Q_\omega) \cup V(e_c)$ is 3-ball. Then $f^k(Q_\omega) \subset B_k \subset int\ f^{k-1}(Q_\omega)$. Thus $B_\omega = f^{1-k}(B_k)$ is the required 3-ball. $\diamond$

3.2 Construction of a quasi-energy function for a diffeomorphism $f \in G_{4,1}$

As a similar construction was done in section 4.3 of [3], we only give a sketch of it below.

1. Construct an energy function $\varphi_p : U_p \to \mathbb{R}$ near each fixed point p of f as in statement 3.1.

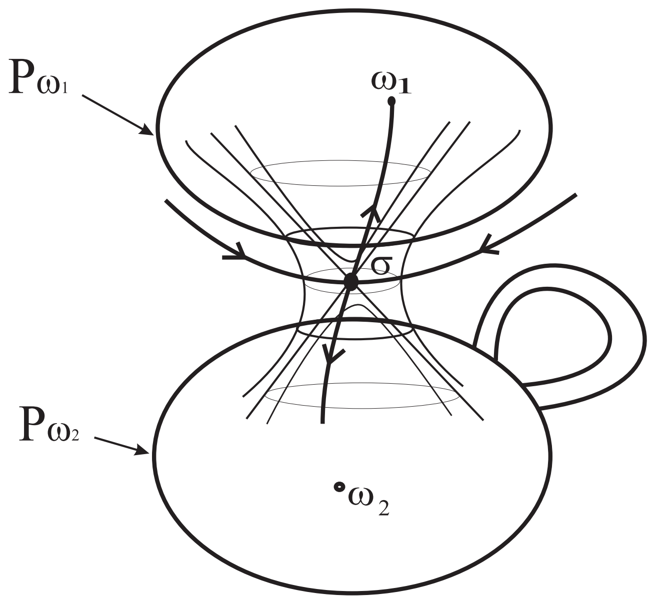

Figure 3: Heegaard decomposition (Q^+, Q^-)

2. By definition of the class $G_{4,1}$, for each $f \in G_{4,1}$ there is a solid torus P^+ belonging to a Heegaard splitting (P^+, P^-) of $\mathbb{S}^3$ and such that:

   a) $\overline{W^u(\sigma)} \subset f(P^+) \subset int\ P^+$;

   b) $W^s(\sigma)$ intersects ∂P^+ transversally and $W^s(\sigma) \cap P^+$ consists of a unique 2-disk.

   As $\mathbb{S}^3 \setminus \overline{W^s(\sigma)}$ is the disjoint union $W^s(\omega_1) \cup W^s(\omega_2)$, then by property b), the disk $P^+ \cap W^s(\sigma)$ is separating in P^+. Moreover there exists a neighborhood of $P^+ \cap W^s(\sigma)$, such that after removing it from P^+ we get a 3-ball P_{ω_1} and solid torus P_{ω_2} with the following properties for each $i = 1, 2$:

   i) $\omega_i \in f(P_{\omega_i}) \subset int\ P_{\omega_i} \subset W^s(\omega_i)$;

   ii) ∂P_{ω_i} is a Heegaard surface and $\ell_i \cap \partial P_{\omega_i}$ consists of exactly one point.

   Due to the λ-lemma[1] (see, for example,[7]), replacing P_{ω_i} by $f^{-n}(P_{\omega_i})$ for some $n > 0$ if necessary, we may assume that ∂P_{ω_i} is transversal to the regular part of the critical level set $C := \varphi_\sigma^{-1}(1)$ of the function φ_σ and the intersections $C \cap \partial P_{\omega_i}$ consist of exactly one circle. For $\varepsilon \in (0, \frac{1}{2})$ define H_ε^+ as the closure of $\{x \in U_\sigma \mid x \notin (P_{\omega_1} \cup P_{\omega_2}),\ \varphi_\sigma(x) \leq 1+\varepsilon\}$ and set $P_\varepsilon^+ = P_{\omega_1} \cup P_{\omega_2} \cup H_\varepsilon^+$. In the same way as in [3] it is possible to choose $\varepsilon > 0$ such that ∂P_{ω_i} intersects transversally each level set with value in $[1-\varepsilon,\ 1+\varepsilon]$; this intersection consists of one circle. Taking a smoothing Q^+ of P_ε^+ we have $f(Q^+) \subset int\ Q^+$ and $\Sigma := \partial Q^+$ is a Heegaard surface of genus 1. Let Q^- be the closure of $\mathbb{S}^3 \setminus int\ Q^+$ (see

[1]The λ-lemma claims that $f^{-n}(S_{\omega_i}) \cap U_\sigma$ tends to $\{x_1 = 0\} \cap U_\sigma$ in the C^1 topology when n goes to $+\infty$.

figure 3). It is easy to check that Q^+ is isotopic to P^+. Therefore, the pair (Q^+, Q^-) is an f-adapted Heegaard splitting with the property that the disk $Q^+ \cap W^s(\sigma)$ lies in U_σ.

3. For each $i = 1, 2$, let $\tilde{P}_{\omega_i}$ be a handlebody of genus $i-1$ such that $f(P_{\omega_i}) \subset \tilde{P}_{\omega_i} \subset int\ P_{\omega_i}$, $\partial\tilde{P}_{\omega_i}$ intersects transversally each level set with value in $[1-\varepsilon, 1+\varepsilon]$ along one circle and $P_{\omega_i} \setminus int\ \tilde{P}_{\omega_i}$ is diffeomorphic to $\partial P_{\omega_i} \times [0,1]$. Define d_i as the closure of $\{x \in U_\sigma \mid x \in (W^s(\omega_i) \setminus \tilde{P}_{\omega_i}),\ \varphi_\sigma(x) = 1-\varepsilon\}$. By construction d_i is a disk whose boundary curve bounds a disk D_i in $\partial\tilde{P}_{\omega_i}$. We form S_i by removing the interior of D_i from $\partial\tilde{P}_{\omega_i}$ and gluing the d_i. Denote $P(S_i)$ the handlebody of genus $i-1$ bounded by S_i and containing ω_i. As in [3] it is possible to choose ε such that $f(P(S_i)) \subset int\ P(S_i)$.

   Let K be the domain between ∂Q^+ and $S_1 \cup S_2$. We introduce T^+, the closure of $\{x \in \mathbb{S}^3 \mid x \notin (P_{\omega_1} \cup P_{\omega_2}),\ 1-\varepsilon \leq \varphi_\sigma(x) \leq 1+\varepsilon\}$; observe $T^+ \subset U_\sigma$. We define a function $\varphi_K : K \to \mathbb{R}$ whose value is $1+\varepsilon$ on ∂Q^+, $1-\varepsilon$ on $S_1 \cup S_2$, coinciding with φ_σ on $K \cap T^+$ and without critical points outside T^+. This last condition is easy to satisfy as the domain in question is a product cobordism. In a similar way to [3], section 4.3, one can check that φ_K is a Morse-Lyapunov function.

4. As $P(S_1)$ is a 3-ball such that $\omega_1 \in f(P(S_1)) \subset int\ P(S_1) \subset W^s(\omega_1)$, then by statement 3.2 there is an energy function $\varphi_{P(S_1)} : P(S_1) \to \mathbb{R}$ for f with S_1 as a level set with value $1-\varepsilon$.

5. As $P(S_2)$ is a solid torus such that $\omega_2 \in f(P(S_2)) \subset int\ P(S_2) \subset W^s(\omega_2)$, then according to lemma 3.3 there is a 3-ball B_{ω_2} such that $f(P(S_2)) \subset B_{\omega_2} \subset int\ P(S_2)$. As in the previous item, there is an energy function $\varphi_{B_{\omega_2}} : B_{\omega_2} \to \mathbb{R}$ for f with ∂B_{ω_2} as a level set with value $\frac{1}{2}$.

6. As $P(S_2)$ is a solid torus, it is obtained from a 3-ball by an orientation reversing identification of a pair of disjoint 2-discs in its boundary; hence the solid torus is the union of a 3-ball and an elementary cobordism of index 1. Since, up to isotopy, there is only one 3-ball in the interior of a solid torus, then $(W_{\omega_2}, \partial B_{\omega_2}, S_2)$ is an elementary cobordism of index 1, where $W_{\omega_2} = P(S_2) \setminus int\ B_{\omega_2}$. Hence W_{ω_2} possesses a Morse function $\varphi_{W_{\omega_2}}$ with only one critical point of index 1 and such that $\varphi_{W_{\omega_2}}(\partial B_{\omega_2}) = \frac{1}{2}$, $\varphi_{W_{\omega_2}}(S_2) = 1-\varepsilon$.

7. Define the smooth function $\varphi^+ : Q^+ \to \mathbb{R}$ by the formula
$$\varphi^+(x) = \begin{cases} \varphi_K(x), x \in K; \\ \varphi_{P(S_1)}(x), x \in P(S_1); \\ \varphi_{B_{\omega_2}}(x), x \in B_{\omega_2}; \\ \varphi_{W_{\omega_2}}(x), x \in W_{\omega_2}. \end{cases}$$
   Then φ^+ is a Morse-Lyapunov function for $f|_{Q^+}$ with one additional critical point.

8. By the construction Q^- is a solid torus such that $\alpha \in f^{-1}(Q^-) \subset int\ Q^- \subset W^u(\alpha)$. Since α is a sink for f^{-1} then, as in item 4, there is a 3-ball B_α such that $f^{-1}(Q^-) \subset B_\alpha \subset int\ Q^-$ and an energy function $\varphi_{B_\alpha} : B_\alpha \to \mathbb{R}$ for f^{-1} with ∂B_α as a level set of value $\frac{1}{2}$.

9. Similarly to item 5, ∂Q^- is obtained from ∂B_α by a surgery of index 1. Therefore $(W_\alpha, \partial Q^-, \partial B_\alpha)$ is an elementary cobordism of index 1, where $W_\alpha = Q^- \setminus int\ B_\alpha$. Hence, W_α possesses a Morse function φ_{W_α} with only one critical point of index 1. We may choose $\varphi_{W_\alpha}(\partial B_\alpha) = \frac{1}{2}$, $\varphi_{W_\alpha}(\partial Q^-) = 2 - \varepsilon$.

10. Define the smooth function $\varphi^- : Q^- \to \mathbb{R}$ by the formula

    $$\varphi^-(x) = \begin{cases} 3 - \varphi_{B_\alpha}(x), x \in \varphi_{B_\alpha}; \\ 3 - \varphi_{W_\alpha}(x), x \in \varphi_{W_\alpha}. \end{cases}$$

    Then φ^- is a Morse-Lyapunov function for $f|_{Q^-}$ with one additional critical point.

11. The function $\varphi : \mathbb{S}^3 \to \mathbb{R}$ defined by $\varphi|_{Q^+} = \varphi^+$ and $\varphi|_{Q^-} = \varphi^-$ is the required Morse-Lyapunov function for the diffeomorphism f with exactly six critical points.

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